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DECREASING REARRANGEMENTS AND DOUBLY STOCHASTIC OPERATORS

BY

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ABSTRACT. In this paper generalizations to measurable functions on a finite measure space (X, Λ, μ) of some characterizations of the Hardy-Littlewood-Pólya preorder relation $\prec$ are considered. Let ρ be a saturated, Fatou function norm such that $L^\infty \subset L^\rho \subset L^1$, and let L^ρ be universally rearrangement invariant. The following equivalence is shown to hold for all $f \in L^\rho$ iff (X, Λ, μ) is nonatomic or discrete: $g \prec f$ iff g is in the ρ-closed convex hull of the set of all rearrangements of f. Finally, it is shown that $g \prec f \in L^1$ iff g is the image of f by a doubly stochastic operator.

1. **Introduction.** In [5] and [6], G. H. Hardy, J. E. Littlewood, and G. Pólya introduced a preorder relation $\prec$ for n-tuples $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$ of real numbers as follows. If $x \in \mathbf{R}^n$ let $x^* = (x_1^*, \cdots, x_n^*)$ denote the point obtained by rearranging the components of x in decreasing order. Then for $x, y \in \mathbf{R}^n$, $y \prec x$ means

$$\sum_{i=1}^{k} y_i^* \le \sum_{i=1}^{k} x_i^*, \quad k = 1, \cdots, n-1,$$

with equality when $k = n$. Hardy, Littlewood, and Pólya characterized this preorder relation as follows [11].

(1.1) **Theorem.** *The following are equivalent for* $x, y \in \mathbf{R}^n$.

(1) $y \prec x$.

(2) $\sum_{i=1}^n \varphi(y_i) \le \sum_{i=1}^n \varphi(x_i)$ *for all continuous convex functions* φ *on* $\mathbf{R}$.

(3) y *is in the convex hull of* $\{z: z^* = x^*\}$.

(4) *There is a doubly stochastic matrix* A *such that* $y = Ax$.

Still another condition equivalent to $y \prec x$ has been given by Muirhead [12] (also see [16]).

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It is of interest to try to generalize this theorem to functions in $L^{\rho}(X, \Lambda, \mu)$, where ρ is a saturated Fatou norm such that $L^{\infty} \subset L^{\rho}$, $L^{\rho'} \subset L^1$ and L^{ρ} is universally rearrangement invariant (u.r.i.). The reader is referred to [9] for a discussion of these notions. A generalization to $L^1[0, 1]$ of (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) has been given by J. V. Ryff ([14], [15]). A generalization of (1) $\Leftrightarrow$ (2) has been given for $\sigma(L^{\infty}, L^1)$ by A. Grothendieck [2]. W. A. J. Luxemburg [9] has independently given a generalization of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) for $\sigma(L^{\rho}, L^{\rho'})$.

After establishing some machinery in §§ 2 and 3, we will in §4 generalize (1) $\Leftrightarrow$ (4) for $L^1(X, \mu)$. Finally in §5 we give a generalization of (1) $\Leftrightarrow$ (3) for the ρ-topology.

2. **Preliminaries.** Let (X, Λ, μ) be a *finite measure space* (m.s.), that is, X is a nonempty point set, Λ is a σ-algebra of subsets of X, μ is a nonnegative countably additive measure on Λ, and $a = \mu(X) < \infty$. We let $\int \cdot \, d\mu$ denote integration over X, and let $M(X, \mu)$ denote the extended real valued measurable functions on X. If $f \in M(X, \mu)$ its *distribution function* is defined by $d_f(s) = \mu(\{x: f(x) > s\})$ for all real s, and its *decreasing rearrangement* by $\delta_f(t) = \inf\{s: d_f(s) \le t\}$ for $0 \le t \le \mu(X)$. The characteristic function of $E \in \Lambda$ is denoted by 1_E, and the decreasing rearrangement of 1_E is denoted by δ_E.

Let (X_1, Λ_1, μ_1) also be a finite m.s. with $\mu_1(X_1) = \mu(X) = a$. A map γ: $X \rightarrow X_1$ is called *measure preserving* (m.p.) if $\mu(\gamma^{-1}[E]) = \mu_1(E)$ for all $E \in \Lambda_1$. If $f \in M(X, \mu)$ and $g \in M(X_1, \mu_1)$, then f and g are called *equimeasurable* (written $f \sim g$) whenever $\delta_f = \delta_g$.

The Hardy-Littlewood-Pólya preorder relation is generalized as follows. If $f^+ \in L^1(X, \mu)$ and $g^+ \in L^1(X_1, \mu_1)$ then $g \prec\prec f$ ("*g is weakly majorized by f*") means $\int_0^t \delta_g \le \int_0^t \delta_f$ for all $0 \le t \le a$, and $g \prec f$ ("*g is majorized by f*") means $g \prec\prec f$ and $\int_0^a \delta_g = \int_0^a \delta_f$.

Finally, $\mathfrak{M}$ denotes the set of all bounded, finitely additive real valued measures ν on Λ such that $\nu(E) = 0$ whenever $\mu(E) = 0$. $\mathfrak{M}$ is known to be a vector lattice, where if $\alpha, \beta \in \mathfrak{M}$, then

$$(\alpha \wedge \beta)(E) = \inf\{\alpha(T) + \beta(T^c \cap E): T \subseteq E, \ T \in \Lambda\}.$$

A measure $0 \le \alpha \in \mathfrak{M}$ is called *purely finitely additive* if the zero measure is the only countably additive measure between 0 and α in the lattice ordering. Every $\nu \in \mathfrak{M}$ can be written $\nu = \nu_c + \nu_p$ where ν_p^+, ν_p^- are purely finitely additive, and ν_c is countably additive [17, Theorem 1.24]. Then $d\nu_c = g_\nu d\mu$ with $g_\nu \in L^1$.

3. **Bounds of some functionals.** Results of later sections depend on the following principle which is a corollary of the Hahn-Banach theorem for a locally convex topological vector space V with continuous dual V^*.

(3.1) **Lemma.** *Let* K *be a closed convex subset of* V *and let* $D \subset K$. *Then* K *is the closed convex hull of* D *iff* $\sup F[D] \geq \sup F[K]$ *for all* $F \in V^*$.

In this section results are given which pave the way for the use of this lemma.

A set $A \in \Lambda$ is called an *atom* of (X, Λ, μ) if $\mu(A) > 0$, and for all $B \in \Lambda$ with $B \subset A$ we have either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. Any measurable function is essentially constant on every atom. A measure space is called *nonatomic* if it has no atoms. Although a nonatomic measure space is not measure-theoretically equivalent to $[0, \mu(X)]$ unless (X, Λ, μ) is separable, these two spaces can be related by a measure preserving map.

(3.2) **Lemma.** *The following are equivalent.*

(1) (X, Λ, μ) *is nonatomic.*

(2) *There is a measure preserving map of* X *into* $[0, \mu(X)[$.

(3) *If* $v - u = \mu(X)$ *then there is a m.p. map of* X *into* $[u, v[$.

(4) *Every right continuous decreasing function on* $[0, \mu(X)]$ *is the decreasing rearrangement of a measurable function on* (X, Λ, μ).

Proof. (1) $\Longrightarrow$ (2). If ϕ is the function in [3, Lemma 7], then $\sigma(x) = \mu(X)\phi(x)$ is measure preserving. Alternatively, we may use [4, 41(2)] to define, for each $u = m/2^n$, $n \geq 0$, $0 \leq m \leq 2^n$, sets B_u such that $\mu(B_u) = u\mu(X)$ and $u < v$ implies $B_u \subset B_v$. Then $\{x: \sigma(x) > s\} = \bigcup \{B_t^c : t > s/a\}$ and we easily compute $\delta_\sigma(t) = \mu(X) - t$. (2) $\Longrightarrow$ (3). If $\sigma: X \longrightarrow [0, a[$ is m.p. and $v - u = a$, then $x \mapsto \sigma(x) + u$ is a m.p. map of X into $[u, v[$. (3) $\Longrightarrow$ (4). Let $\sigma: X \longrightarrow [0, a[$ be m.p. If F is decreasing and right-continuous on $[0, a]$ then $f = F \circ \sigma \sim F$, so $\delta_f = F$ by uniqueness of δ_f. (4) $\Longrightarrow$ (1). Let $f \in M(X, \mu)$ such that $\delta_f(t) = a - t$. Then f is not constant on any subset of X of positive measure so X has no atoms.

Let (X, Λ, μ) be nonatomic and let $f \in M(X, \mu)$. If $A, B \in \Lambda$ have $\mu(A) = \mu(B)$, then (3.2) may be used to define $f' =$ *a result of interchanging the values of* f *on* A *and* B, as follows. Let $\sigma_A: A \longrightarrow [0, \mu(A)]$ and $\sigma_B: B \longrightarrow [0, \mu(B)]$ be m.p. Then $f' = \delta_{f|A} \circ \sigma_B$ on B, $= \delta_{f|B} \circ \sigma_A$ on A, $= f$ elsewhere. Clearly $f' \sim f$.

Using (3.2) it is also easy to generalize to nonatomic m.s. a result of J. V. Ryff [15, Lemma 2] and G. Lorentz [7, p. 61] for $[0, 1]$.

(3.3) **Proposition** **(Lorentz-Ryff).** *If the finite m.s.* (X, Λ, μ) *is nonatomic and* $f \in M(X, \mu)$ *then there is a measure preserving map* $\sigma: X \longrightarrow [0, \mu(X)[$ *such that* $f = \delta_f \circ \sigma$ μ*-a.e.*

Proof. See [1, p. 26].

The next result is a generalization proved in [9, p. 102] of an inequality of Hardy and Littlewood.

(3.4) **Lemma.** *If* $f, g \in M(X, \mu)$, $a = \mu(X) < \infty$, *and* $\delta_{|f|}\, \delta_{|g|} \in L^1[0, a]$ *then* $fg \in L^1(X, \mu)$ *and*

$$\int_0^a \delta_f(a - t)\delta_g(t)\,dt \le \int fg\, d\mu \le \int_0^a \delta_f \delta_g.$$

These inequalities hold also for all $0 \le f, g \in M(X, \mu)$, *even if* $\delta_f\delta_g \notin L^1[0, a]$.

It is a corollary of a theorem of Hardy [9, p. 94] that if $f' \prec\prec f$ and both $\delta_{|f|}\,\delta_{|g|}$ and $\delta_{|f'|}\,\delta_{|g|} \in L^1[0, a]$, then

$$\int_0^a \delta_{|f'|}\delta_{|g|} \le \int_0^a \delta_{|f|}\delta_{|g|}.$$

If $f' \prec f$ then in addition

$$\int_0^a \delta_f(t)\delta_g(a - t)\,dt \le \int f'g\, d\mu \le \int_0^a \delta_f\delta_g. \tag{3.5}$$

If $f' \prec f \in L^1(\mu)$ and $\delta_{|f|}\,\delta_{|g|} \in L^1[0, a]$ then, by approximating $|g|$ by nonnegative simple functions, we see that already $\delta_{|f'|}\,\delta_{|g|} \in L^1[0, a]$, and (3.5) holds.

Because of its utility, Luxemburg has called a measure space *adequate* if $\max\{\int fg'\, d\mu : g' \sim g\} = \int_0^a \delta_f\, \delta_g$ for all $0 \le f, g \in M(X, \mu)$, and he has asked for a characterization of such measure spaces [9, p. 106]. The following seems to be "adequate".

(3.6) **Theorem.** *The following are equivalent for the finite m.s.* (X, Λ, μ).

(1) (X, Λ, μ) *is adequate.*

(2) (X, Λ, μ) *is nonatomic or consists only of atoms of equal measure.*

(3) *For all* $A, B \in \Lambda$ *we have*

$$\sup\left\{\int 1_A 1_E\, d\mu : 1_E \sim 1_B\right\} = \sup\{\mu(A \cap E) : \mu(E) = \mu(B)\} = \int_0^a \delta_A\delta_B.$$

Proof. (2) $\Rightarrow$ (1). Suppose (X, Λ, μ) is nonatomic. Let $\sigma: X \to [0, a]$ be m.p. such that $\delta_f \circ \sigma = f$ μ-a.e. Then $\int_0^a \delta_f\, \delta_g = \int (\delta_f \circ \sigma)(\delta_g \circ \sigma)\, d\mu = \int fg'd\mu$, where $g' = \delta_g \circ \sigma \sim g$. The proof when (X, μ) is discrete is similar [5, Theorem 368]. (1) $\Rightarrow$ (3) is obvious. It remains to prove (3) $\Rightarrow$ (2). Suppose (2) is not true. Then either X has at least two atoms, A, B with $0 < \mu(B) < \mu(A)$; or X has an atom A and a nonatomic part X_0 of positive measure, in which case there is a $B \subset X_0$ such that $0 < \mu(B) < \mu(A)$. In either case, for all $E \in \Lambda$ with $1_E \sim 1_B$ we have $\mu(E) = \mu(B)$ and hence $\mu(A \cap E) \le \mu(E) = \mu(B) < \mu(A)$, so $\mu(A \cap E) = 0$, but $\int_0^a \delta_A\, \delta_B = \mu(B) > 0$.

Finally, it is necessary to determine $\sup\{\int h\, d\nu : h \sim f\}$ when $f \in L^\infty$, $\nu \in \mathfrak{M}$, and (X, Λ, μ) is nonatomic.

(3.7) **Lemma.** *Suppose* $0 < \alpha$, $\beta \in \mathfrak{M}$ *are purely finitely additive. If* $\alpha \wedge \beta = 0$ *then there are sequences* $\{A_n\}$ *and* $\{B_n\}$ *such that*

(a) $A_n \cap B_n = \emptyset$, (c) $\beta(B_n) \uparrow \beta(X)$,

(b) $\alpha(A_n) \uparrow \alpha(X)$, (d) $\mu(A_n)$ *and* $\mu(B_n) \to 0$.

The proof is straightforward using [17, 1.1.1 and Theorem 1.22].

(3.8) **Lemma.** *Suppose* (X, Λ, μ) *is nonatomic,* $A \cap B = \emptyset$ *and* $\mu(A)$, $\mu(B) \leq \frac{1}{4} \min\{\mu(S), \mu(T)\}$. *Then there are sets* $D \subset S$ *and* $E \subset T$ *such that*

(a) $\mu(D) = \mu(A), \mu(E) = \mu(B)$;

(b) *A, B, D and E are pairwise disjoint.*

Proof. $\mu(A) + \mu(A^c \cap B \cap S)/2 + \mu(A \cap B^c \cap S)/2 \leq \mu(A) + \mu(B)/2 + \mu(A)/2 \leq \mu(S)/4 + \mu(S)/8 + \mu(S)/8 = \mu(S)/2$. Hence, $\mu(A) \leq \mu(A^c \cap B^c \cap S)/2 = [\mu(A^c \cap B^c \cap S \cap T^c) + \mu(A^c \cap B^c \cap S \cap T)]/2$. Similarly, $\mu(B) \leq [\mu(A^c \cap B^c \cap S \cap T) + \mu(A^c \cap B^c \cap S^c \cap T)]/2$. Since (X, Λ, μ) is nonatomic, $A^c \cap B^c \cap S \cap T = P \cup Q$ with $P \cap Q = \emptyset$ and $\mu(P) = \mu(Q)$. Hence $\mu(A) \leq \mu([A^c \cap B^c \cap S \cap T^c] \cup P)$ so there is a $D \subset (A^c \cap B^c \cap S \cap T^c) \cup P$ such that $\mu(D) = \mu(A)$. Similarly for $E \subset (A^c \cap B^c \cap S^c \cap T) \cup Q$.

(3.9) **Proposition.** *Suppose* (X, Λ, μ) *is nonatomic, let* $\nu \in \mathfrak{M}$ *and* $f \in L^\infty$. *Then*

$$\sup\left\{\int h\, d\nu : h \sim f\right\} = \int_0^a \delta_f \delta_{g_\nu} + \nu_p^+(X) \text{ ess sup } f - \nu_p^-(X) \text{ ess inf } f.$$

Proof. Let $r = \text{ess sup } f$, $s = \text{ess inf } f$, let $\sigma: X \to [0, a]$ be m.p. such that $\delta_{g_\nu} \circ \sigma = g_\nu$ μ-a.e., let $h = \delta_f \circ \sigma$, and for $i \geq 1$ let $S_i = \{|f - r| < 1/i\}$ and $T_i = \{|f - s| < 1/i\}$. Let $\{A_n\}$ and $\{B_n\}$ satisfy (a)–(d) in (3.7) with $\alpha = \nu_p^+$ and $\beta = \nu_p^-$. Using (d) and passing to subsequences if necessary, we may assume $\mu(A_i), \mu(B_i) \leq \frac{1}{4} \min\{\mu(S_i), \mu(T_i)\}$. Hence by (3.8) there are sets $D_i \subset S_i$ and $E_i \subset T_i$ such that $\mu(D_i) = \mu(A_i)$, $\mu(E_i) = \mu(B_i)$, and A_i, B_i, D_i, E_i are pairwise disjoint.

For each $i \geq 1$ let h_i be a result of first interchanging the values of h on D_i and A_i, and then of interchanging the values of the resulting function on E_i and B_i. Then $h_i \sim h \sim f$, and $\int h_i d\nu = \int h_i g_\nu d\mu + \int h_i d\nu_p^+ - \int h_i d\nu_p^-$. Now $\int h_i g_\nu d\mu \to \int h g_\nu d\mu$ as $i \to \infty$. Indeed, if $G_i = A_i \cup B_i \cup D_i \cup E_i$ then $h_i = h$ on $X \setminus G_i$, so

$$\int h_i g_\nu d\mu = \int_{X - G_i} h g_\nu d\mu + \int_{G_i} h_i g_\nu d\mu, \quad \left|\int_{G_i} h_i g_\nu d\mu\right| \leq \|h\|_\infty \int_{G_i} |g_\nu|\, d\mu \to 0$$

as $i \to \infty$. For the rest,

$$\left|\int h_i d\nu_p^+ - r\nu_p^+(X)\right| \le \int_{A_i} |h_i - r|\, d\nu_p^+ + \int_{X-A_i} |h_i - r|\, d\nu_p^+$$
$$\le \frac{1}{i}\, \nu_p^+(X) + \|h - r\|_\infty [\nu_p^+(X) - \nu_p^+(A_i)] \longrightarrow 0.$$

Similarly $|\int h_i\, d\nu_p^- - s\nu_p^-(X)| \to 0$. Since $\int hg_\nu\, d\mu = \int (\delta_f \circ \sigma)(\delta_{g_\nu} \circ \sigma) d\mu = \int_0^a \delta_f \delta_{g_\nu}$, the proof is finished.

4. **Doubly stochastic operators.** If $T: L^1(\mu_1) \to L^1(\mu)$ is bounded and linear, let T^* denote the *adjoint* of T, defined by $\int g\, Tf\, d\mu = \int f T^* g\, d\mu_1$, for all $f \in L^1(\mu_1)$ and $g \in L^\infty(\mu)$. Then $T^*: L^\infty(\mu) \to L^\infty(\mu_1)$ and T is weakly continuous [13, p. 38, Proposition 12] (or use nets and the defining equation).

By analogy with the definition for matrices, we define a *doubly stochastic* (d.s.) operator to be a bounded, linear operator $T: L^1(\mu_1) \to L^1(\mu)$ such that (1) $T \ge 0$; (2) $T1_{X_1} = 1_X$; and (3) $T^* 1_X = 1_{X_1}$. It is easy to see that whenever two d.s. operators can be composed, the result is d.s.

(4.1) **Theorem.** *Let T be a linear map of the simple functions of $L^1(\mu_1)$ into $L^1(\mu)$. The following are equivalent*:

(1) *T extends to a d.s. operator on $L^1(\mu_1)$.*

(2) *$0 \le T1_E \le 1_X$ and $\int T1_E\, d\mu = \mu_1(E)$ for all $E \in \Lambda_1$.*

(3) *There is a linear extension of T to $L^1(\mu_1)$ such that $Tf \prec f$ for all $f \in L^1(\mu_1)$.*

In (1) *and* (3) *the extension is necessarily unique.*

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3) is proved as in [9, p. 130, (ii)]. For (3) $\Rightarrow$ (1), prove $T \ge 0$ as in [14, p. 1381], and prove $T1_{X_1} = 1_X$ as in [9, (6.2. iii)]. To show $T^*1_X = 1_{X_1}$ is easy. To see that the extensions are unique, note that $T \ge 0$ implies $(Tf)^+ \le Tf^+$ and $(Tf)^- \le Tf^-$, so T is a contraction in both the L^1 and L^∞ norms.

Remark. (i) (2) $\Leftrightarrow$ (3) was first proved by J. V. Ryff for $L^1[0, 1]$.

(ii) (1) $\Rightarrow$ (3) here generalizes (1.1) (4) $\Rightarrow$ (1).

(4.2) **Proposition.** *If $T: L^1(\mu_1) \to L^1(\mu)$ is d.s. then T^* has a unique extension to a d.s. map of $L^1(\mu) \to L^1(\mu_1)$.*

Proof. We verify (4.1.2) for T^*. Let $E \in \Lambda$. For all $A \in \Lambda_1$, $\int_A T^*1_E d\mu_1 = \int 1_E T1_A d\mu$, and $0 \le \int 1_E T1_A\, d\mu \le \int T1_A\, d\mu = \mu_1(A) = \int_A 1_{X_1} d\mu_1$, so $0 \le T^*1_E \le 1_{X_1}$. The rest is easy.

(4.3) **Example.** T_γ. If $\gamma: X \to X_1$ is m.p., define $T_\gamma f = f \circ \gamma$ for all $f \in L^1(\mu_1)$. Then $T_\gamma f \sim f$, so T_γ is d.s. Of more importance, $T_\gamma^* T_\gamma f = f$ for all $f \in L^1(\mu_1)$. Indeed, for all $A \in \Lambda_1$ and $f \in L^1(\mu_1)$, $\int 1_A T_\gamma^* T_\gamma f d\mu_1 = \int T_\gamma 1_A T_\gamma f d\mu = \int (1_A \circ \gamma)(f \circ \gamma) d\mu = \int 1_A f d\mu_1$ (see [15, Lemma 3]).

(4.4) **Example.** T_μ [9, p. 99]. The conditional expectation operator determined by a σ-subalgebra of Λ is easily shown to be d.s. using (4.1). An important example is the d.s. operator T_μ which arises when a finite m.s. is embedded in a nonatomic m.s. Note that a finite or σ-finite m.s. has at most countably many atoms, so $X = X_0 \cup \bigcup_{n \in P} A_n$, where X_0 is nonatomic, each A_n is an atom, $P = \{1, \cdots, k\}$ or $P = \{1, 2, 3, \cdots\}$, and $\mu(A_i \cap A_j) = 0$ whenever $i \neq j$.

We embed (X, Λ, μ) in a nonatomic m.s. $(X^\#, \Lambda^\#, \mu^\#)$ as follows. Let $I[a_n, b_n]$ be disjoint intervals of $\mathbf{R}$ with endpoints a_n and b_n, such that $b_n - a_n = \mu(A_n)$, $n \in P$. Then define

$$X^\# = X_0 \cup \bigcup_{n \in P} I[a_n, b_n];$$

$E \in \Lambda^\#$ iff $E = E_0 \cup \bigcup_{n \in P} J_n$ where $E_0 \in X_0 \cap \Lambda$ and $J_n \subset I[a_n, b_n]$ is Lebesgue measurable; and $\mu(E) = \mu(E_0) + \Sigma_{n \in P}\, m(J_n)$ where m = Lebesgue measure. Each $f \in M(X, \mu)$ is identified with $f^\# = f 1_{X_0} + \Sigma_{n \in P} (f|A_n) 1_{I[a_n, b_n]}$. Clearly $f^\# \sim f$, so $\delta_{f^\#} = \delta_f$.

Finally, we define

$$T_\mu f = f 1_{X_0} + \sum_{n \in P} \left(\frac{1}{b_n - a_n} \int_{a_n}^{b_n} f \right) 1_{A_n} \qquad \text{for all } f \in M(X^\#, \mu^\#)$$

for which this makes sense. Then:

(1) $T_\mu: L^1(\mu^\#) \to L^1(\mu)$ is d.s.;

(2) for all $f \in L^1(\mu^\#)$ and $g \in M(X, \mu)$ such that $fg^\# \in L^1(\mu^\#)$ we have $T_\mu(g^\# f) = g T_\mu f$, so $T_\mu g^\# = g$ and $\int g^\# f d\mu^\# = \int g T_\mu f d\mu$.

We now give a generalization of (1.1) (1) $\Rightarrow$ (4). Let $\mathfrak{D}(X_1, X) = \{T \mid T: L^1(\mu_1) \to L^1(\mu)$ is d.s.$\}$, $\mathfrak{D}_f(X_1, X) = \{Tf : T \in \mathfrak{D}(X_1, X)\}$, and $\Omega_f(X, \mu) = \{g \in M(X, \mu): g \prec f\}$ for $f \in L^1(\mu_1)$. Usually we will abbreviate these sets as $\mathfrak{D}$, $\mathfrak{D}_f$, Ω_f, respectively.

(4.8) **Lemma (Ryff).** *$\mathfrak{D}(X_1, X)$ is convex and compact in the weak operator topology determined by the linear functionals $T \mapsto \int f T g\, d\mu$, $f \in L^1(\mu)$, $g \in L^\infty(\mu_1)$.*

The proof given in [15, p. 97] generalizes easily.

(4.9) **Theorem.** *Let $f \in L^1(X_1, \mu_1)$. If $g \in M(X, \mu)$ then $g \prec f$ iff there is a doubly stochastic operator $T: L^1(\mu_1) \to L^1(\mu)$ such that $g = Tf$.*

Proof. Let $f \in L^1(\mu_1)$. Clearly $\mathfrak{D}_f \subset \Omega_f$, so it suffices to show that $\Omega_f \subset \mathfrak{D}_f$. Now $\mathfrak{D}_f$ is a convex, weakly closed subset of $L^1(\mu)$, because it is the image of the compact, convex set $\mathfrak{D}(X, X_1)$ under the continuous, linear map $T \mapsto T^* f$. Letting $K = \overline{\text{cov}}\, \Omega_f$ ($= \Omega_f$, actually, but we do not need this), it suffices to show $K = \overline{\text{cov}}\, \mathfrak{D}_f = \mathfrak{D}_f$. We do this using Lemma (3.1). Let $g \in L^\infty(\mu)$. Then

$$\sup\left\{\int gh\, d\mu:\ h \in K\right\} = \sup\left\{\int gh\, d\mu:\ h \in \Omega_f\right\} \leq \int_0^a \delta_f \delta_g = \int f' g^\# \, d\mu^\# = \int g T_\mu f' \, d\mu$$

for some $f' \in M(X^\#, \mu^\#)$ such that $f' \sim f$. Let $\sigma: X^\# \to [0, a]$ and $\gamma: X_1^\# \to [0, a]$ be measure preserving such that $T_\sigma \delta_{f'} = f'$ and $T_\gamma \delta_f = f$. Since $f' \sim f$, $\delta_{f'} = \delta_f$, so $T_\mu f' = T_\mu T_\sigma \delta_{f'} = T_\mu T_\sigma \delta_f = T_\mu T_\sigma T_\gamma^* f \in \mathfrak{D}_f$, and the proof is finished.

(4.10) **Corollary**. *If $f_1, f_2 \in L^1(X_1, \mu_1)$ and $g \in M(X, \mu)$ and $g \prec f_1 + f_2$ then there are $g_1, g_2 \in L^1(X, \mu)$ such that $g = g_1 + g_2$ and $g_1 \prec f_1$ and $g_2 \prec f_2$.*

This generalizes [8, p. 51].

5. **$\Omega(f)$ is the closed convex hull of $\Delta(f)$.** If $f \in L^1(X, \mu)$ let $\Delta(f) = \{h \in M:\ h \sim f\}$ and $\Omega(f) = \{h \in M:\ h \prec f\}$. One way to generalize (1.1) (1) $\Leftrightarrow$ (3) is to give conditions on a Banach function space B between L^∞ and L^1 such that for all $f \in B$, $\Omega(f)$ is the norm closed convex hull of $\Delta(f)$. That $\Omega(f)$ is convex when $f \in L^1$ follows as in [9, p. 135].

We will consider the class of Banach function spaces L^ρ described in detail in [9], and [10]. Recall that a *Riesz function norm* is a mapping $\rho: M^+(X, \mu) \to [0, \infty]$ which is zero only at functions which are zero μ-a.e., which is positive homogeneous, satisfies the triangle inequality, and which is increasing: $0 \leq f \leq g$ implies $\rho(f) \leq \rho(g)$. The norm ρ is said to be *Fatou* if $0 \leq f_n \uparrow f$ pointwise implies $\rho(f_n) \uparrow \rho(f)$. We extend ρ to $M(X, \mu)$ be defining $\rho(f) = \rho(|f|)$ and let $L^\rho(X, \mu)$ denote those f for which $\rho(f) < \infty$. If $A \in \Lambda$ implies there is a $B \in \Lambda$ with $B \subset A$, $\mu(B) > 0$, and $\rho(1_B) < \infty$, then ρ is said to be *saturated.* Associated with ρ are ρ' and ρ'' defined by $\rho'(f) = \sup\{\int |fg|\, d\mu:\ \rho(g) \leq 1\}$ and $\rho'' = (\rho')'$.

We assume for the remainder of this section that ρ is a saturated Fatou function norm such that $L^\infty \subset L^\rho \subset L^1$. Then L^ρ is complete [10, Note II, p. 149] and $L^\infty \subset L^{\rho'} \subset L^1$. We also assume that $\Omega(f) \subset L^\rho$ whenever $f \in L^\rho$, which is equivalent to having $\delta_{|f|}\, \delta_{|g|} \in L^1[0, a]$ whenever $f \in L^\rho$ and $g \in L^{\rho'}$ [9, p. 116]. Such spaces L^ρ are called (u.r.i.) by Luxemburg. It follows as in [9, pp. 135, 136] that $\Omega(f)$ is ρ-closed and ρ-bounded for all $f \in L^\rho$.

(5.1) **Proposition**. *If $L^\rho \neq L^\infty$, then $(L^\rho)^* = L^{\rho'}$.*

Proof. Now $(L^\rho)^* \subset (L^\infty)^* \approx \mathfrak{M}$. Suppose $f \in L^\rho \setminus L^\infty$, and let $\nu \in (L^\rho)^*$. Since $|\nu_p| \wedge |\nu_c| = 0$ [17, Theorem 1.16], we have $|\nu_c| + |\nu_p| = |\nu| \in (L^\rho)^*$ [10, Note VII, Theorem 22.3], so $|\nu_p| \in (L^\rho)^*$ [10, Note VII, Theorem 22.4]. Now $|\nu_p| \ll \mu$, so every atom of μ is an atom of $|\nu_p|$. Then $|\nu_p|$ is both countably and purely finitely additive on each atom of μ, so $|\nu_p| = 0$ on the atoms of μ, if any. If $\mu(X_0) > 0$, where X_0 is the nonatomic part of X, and $|\nu_p|(X_0) \neq 0$, then the proof of (3.9) shows that $\sup\{|\int h\,d|\nu_p|: h\sim |f|\} = +\infty$, which contradicts $|\nu_p| \in (L^\rho)^*$ and ρ-boundedness of $\Omega(f)$. Thus $|\nu_p|(X_0) = 0$, so $d\nu = g_\nu d\mu$, and $g_\nu \in L^{\rho'}$ by the converse of Hölder's inequality [10, Note V, Theorem 14.1].

(5.2) **Theorem.** *$\Omega(f)$ is the ρ-closed convex hull of $\Delta(f)$ for all $f \in L^\rho$ iff (X, Λ, μ) is adequate.*

Proof. If $f \in L^\rho$, then Lemma (3.1) says that $\Omega(f)$ is the closed convex hull of $\Delta(f)$ iff

$$(*) \qquad \sup F[\Delta(f)] \geq \sup F[\Omega(f)] \qquad \text{for all } F \in (L^\rho)^*.$$

If (X, Λ, μ) is not adequate, then Theorem (3.6) says there are $A, B \in \Lambda$ such that

$$\sup\left\{\int 1_A 1_E\,d\mu: 1_E \sim 1_B\right\} = 0 < \mu(B) = \int_0^a \delta_A\delta_B$$
$$= \sup\left\{\int 1_A T_\mu 1_E\,d\mu: 1_E \sim 1_B,\ E \in \Lambda^\#\right\} \leq \sup\left\{\int 1_A h\,d\mu: h \prec 1_B\right\}.$$

Since $(L^\rho)^* \supset L^\infty$, $(*)$ fails for $f = 1_A$ and $F(\cdot) = \int \cdot\, 1_B\,d\mu$.

Conversely, suppose (X, Λ, μ) is adequate and let $f \in L^\rho$. If (X, Λ, μ) is discrete, then Theorem (1.1) gives the result. Thus let (X, Λ, μ) be nonatomic. Suppose first that $L^\rho \neq L^\infty$, so $(L^\rho)^* = L^{\rho'}$. Let $g \in L^{\rho'}$, and let $\sigma: X \to [0, a]$ be m.p. such that $\delta_g \circ \sigma = g\mu$-a.e. Since $\delta_{|f|}\,\delta_{|g|} \in L^1[0, a]$ we have

$$\sup\left\{\int hg\,d\mu: h \in \Omega(f)\right\} \leq \int_0^a \delta_f\delta_g = \int (\delta_f \circ \sigma) g\,d\mu,$$

so $(*)$ holds. If $L^\rho = L^\infty$, and $\nu \in \mathfrak{M}$, then

$$\sup\left\{\int h\,d\nu: h \in \Omega(f)\right\} \leq \int_0^a \delta_f\delta_{g_\nu} + \nu_p^+(X)\ \text{ess sup}\, f - \nu_p^-(X)\ \text{ess inf}\, f,$$

so Proposition (3.9) shows that $(*)$ holds.

6. **Problem.** The following problem, suggested by the previous results, seems to be open:

Let (X, Λ, μ) be nonatomic, let $f \in L^\infty$, and $\nu \in \mathfrak{M}$. Characterize $\{\int h\,d\nu: h \sim f\}$ as a subset of **R**. For example, [9, Theorem (9.1)] and the proof of Proposition (3.9) show that this set is dense in

$$\left[\int_0^a \delta_f(t)\delta_{g_\nu}(a-t), \int_0^a \delta_f\delta_{g_\nu}\right] + \{r\nu_p^+(X) - s\nu_p^-(X): r, s \in R_f\}$$

where R_f is the essential range of f.

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